Barker Student Number:________________

Barker College

Mathematics Extension 1

2010 TRIAL HIGHER SCHOOL CERTIFICATE

PM FRIDAY 13 AUGUST

Staff Involved:

- GDH*
- LJP*
- BHC
- WMD
- TRW
- GIC
- VAB

85 copies

General Instructions

- Reading time – 5 minutes
- Working time – 2 hours
- Write using blue or black pen
- Make sure your Barker Student Number is on ALL pages
- Board-approved calculators may be used
- A table of standard integrals is provided on page 13
- ALL necessary working should be shown in every question
- Marks may be deducted for careless or badly arranged working

Total marks – 84

- Attempt Questions 1 – 7
- All questions are of equal value

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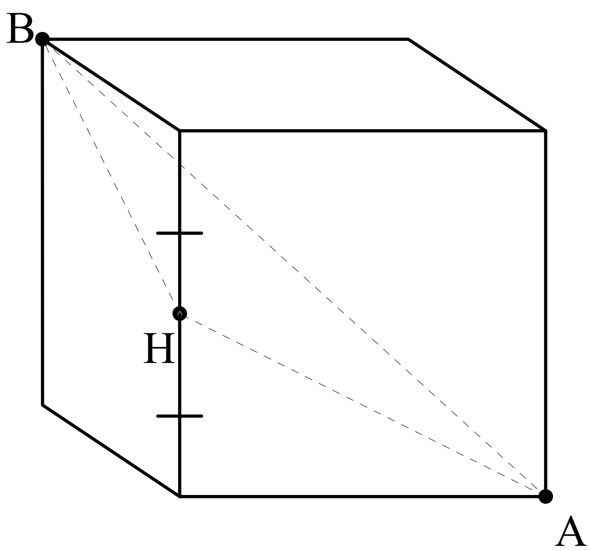
Question 1 (12 marks) [BEGIN A NEW PAGE]
Marks

(a) Solve $\dfrac{x - 3}{x - 2} < 1$ **2**

(b) The diagram shows a cube with edge length 2 units.

H is the midpoint of the edge shown.

Using triangle AHB or otherwise, find the size of $\angle AHB$ **3**



(c) (i) Write as a single fraction: $\dfrac{1}{x + h} - \dfrac{1}{x}$ **1**

(ii) Using $f'(x) = \lim_{h \to 0} \dfrac{f(x + h) - f(x)}{h}$,

find $f'(x)$ from first principles if $f(x) = \dfrac{1}{x}$ **2**

(d) Find $\int_{-3}^{0} x\sqrt{1 - x}\, dx$ using the substitution $u = 1 - x$ **4**

Question 2 (12 marks) [BEGIN A NEW PAGE]
Marks

(a) The region under the curve $y = \cos 3x$ between $x = \frac{\pi}{18}$ and $x = \frac{\pi}{9}$ **4**

is rotated about the x-axis.

Evaluate the volume of the solid of revolution formed.

(b) Consider $f(x) = x^6 - 2x^4 + x^2$

(i) Explain why $f(x)$ is an even function **1**

(ii) Factorise $x^4 - 2x^2 + 1$ **1**

(iii) Find the three x-intercepts of $f(x)$ **1**

For the remainder of the question, use part (i) to save time.

(iv) Find the coordinates of the five stationary points of $f(x)$ **3**

(v) Determine the nature of the stationary points **1**

(vi) Sketch the graph of $y = f(x)$. Do not find points of inflexion. **1**

Question 3 (12 marks) **[BEGIN A NEW PAGE]**
Marks

(a) (i) Sketch $y = (\cos^{-1} x) - \dfrac{\pi}{2}$ showing all important features **2**

(ii) Differentiate $y = x\cos^{-1} x - \sqrt{1-x^2}$ **2**

(iii) Hence or otherwise evaluate $\displaystyle\int_{-1}^{1} (\cos^{-1} x) - \frac{\pi}{2}\, dx$ **2**

(b) The acceleration of a particle moving along a straight path is given by

$\ddot{x} = (\ln 3)^2 x$

Initially, $x = 1$ and $v = \ln 3$

(i) Show that $v = (\ln 3)x$ **3**

(ii) Find x as a function of t **3**

Question 4 (12 marks) [BEGIN A NEW PAGE]
Marks

(a) Consider $f(x) = 2^{-x} + 1$

(i) Draw a large sketch $y = f(x)$ showing all important features **1**

(ii) Find $f^{-1}(x)$ **2**

(iii) Using parts (i) or (ii),
accurately sketch $y = f^{-1}(x)$ on the same set of axes as used in part (i) **1**

(b) A particle is moving in a straight line and its position x is given by the equation
$x = \cos\left(3t - \frac{\pi}{2}\right)$ where x is the displacement in metres and t is the time in seconds.

(i) Prove that the particle is moving in Simple Harmonic Motion **2**

(ii) Sketch the displacement of the particle for the domain $0 \le t \le \pi$ showing all important features **2**

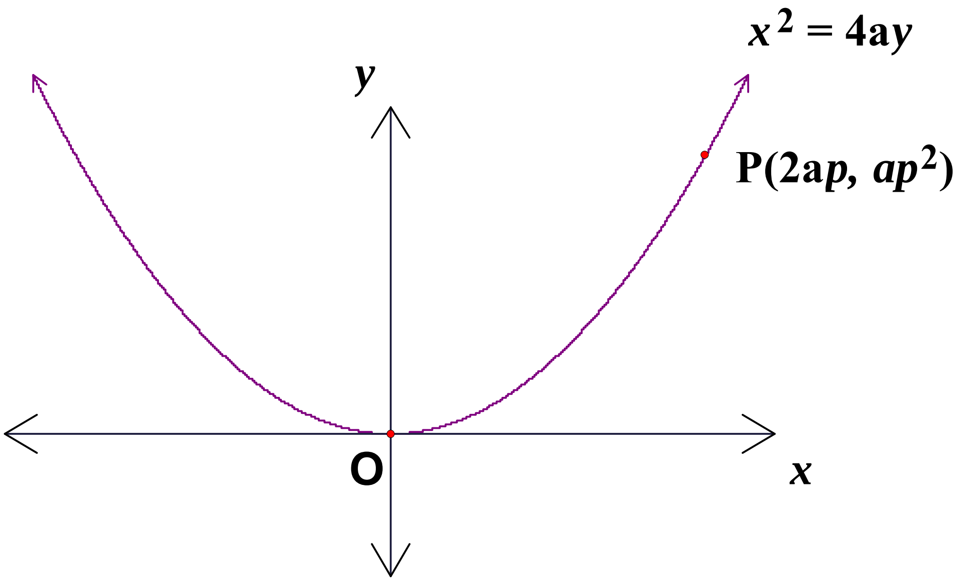
(iii) How far did the particle travel in the first π seconds? **1**

(iv) The displacement is currently defined in terms of cosine.
Define the displacement in terms of sine. **1**

(v) When is the first time that the particle is travelling at half its maximum speed? **2**

Question 5 (12 marks) [BEGIN A NEW PAGE]
Marks

(a)



(i) Find the gradient of the line **perpendicular** to OP **2**

(ii) The point A is on the x-axis. AP is parallel to the y-axis.
State the coordinates of the point A **1**

(iii) Find the equation of the line through A perpendicular to OP **1**

(iv) Through what point does the line in (iii) always pass? **1**

Question 5 continues on the next page

Question 5 (continued)
Marks

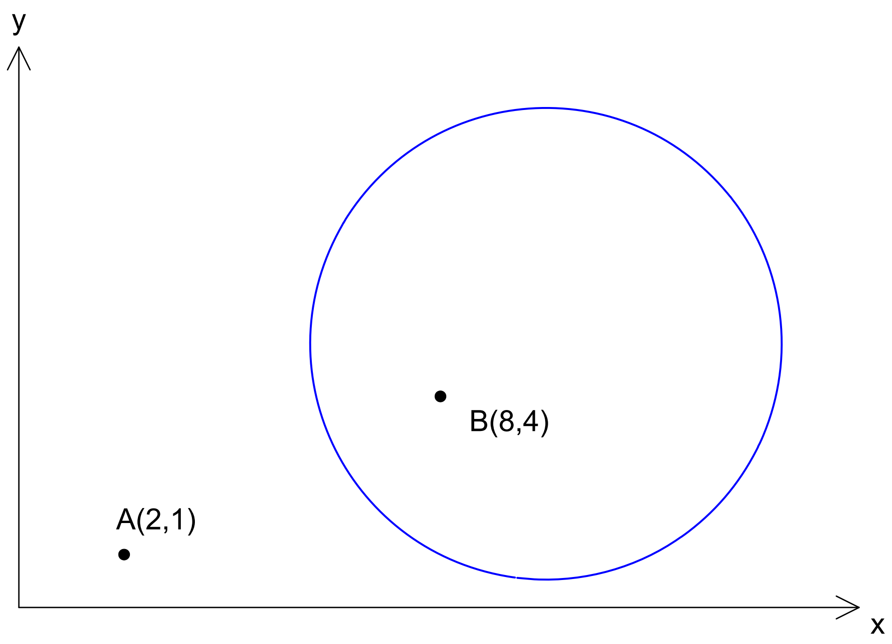
(b) Consider the points A(2,1) and B(8,4)

(i) The point I divides the interval AB **internally** in the ratio 2:1.
Find the coordinates of the point I **1**

(ii) The point E divides the interval AB **externally** in the ratio 2:1.
Find the coordinates of the point E **2**

A point P moves such that it is always twice as far from A(2,1) as it is from B(8,4).

The locus of P is the circle as shown below.



(iii) Using parts (i) and (ii) or otherwise, find the equation of the locus of P.

Your equation must be in the form where the centre and radius are easily determined. **4**

Question 6 (12 marks) [BEGIN A NEW PAGE]
Marks

(a) (i) How many six letter 'words' can be made from the letters of the word

CHANCE? **1**

(ii) What is the probability that one of the words in part (i) contains the word

EACH? **2**

(iii) How many three letter 'words' can be made from the letters of the word

CHANGE? **1**

(iv) How many three letter 'words' can be made from the letters of the word

CHANCE? **2**

(b) Find the simplified sixth term of the binomial expansion of $\left(\frac{2x}{3}-\frac{3}{2x}\right)^{11}$ **2**

(c) Prove, by mathematical induction, the formula for the sum of an arithmetic series.

i.e. Prove, by mathematical induction, that

$a+[a+d]+[a+2d]+...+[a+(n-1)d]=\frac{n}{2}\left[2a+(n-1)d\right]$ for all integers $n\geq 1$ **4**

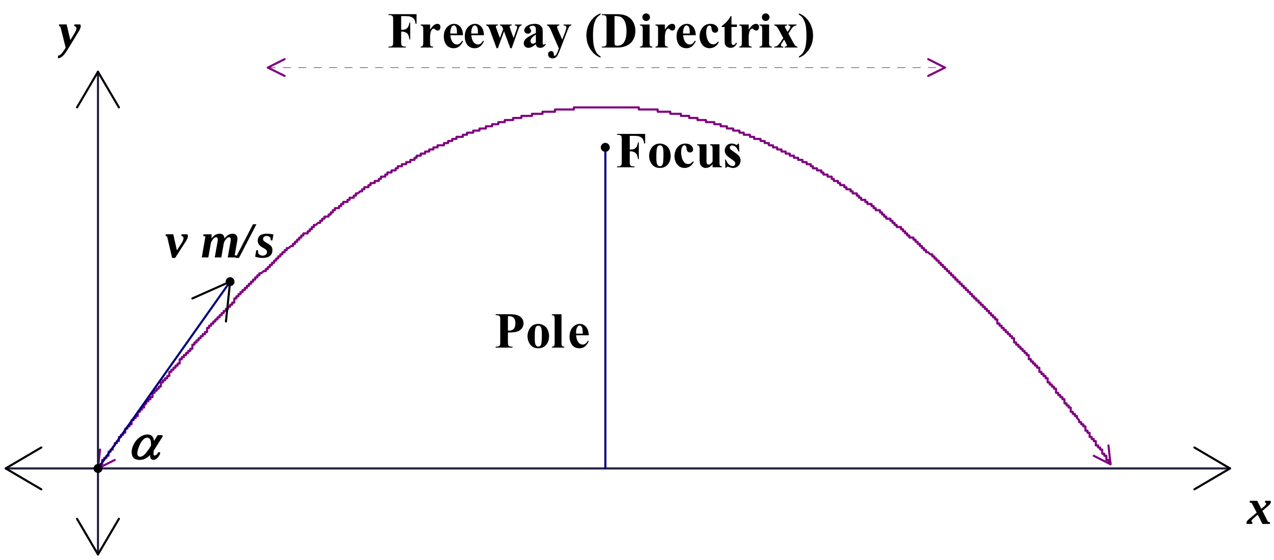
Question 7 (12 marks) [BEGIN A NEW PAGE]
Marks

(a) A particle moves on a curve with equation $y = 4 + 2\tan^{-1}(3x^2 + 5)$.

$\frac{dx}{dt}$ is a finite constant.

What is the limiting value of $\frac{dy}{dt}$ as $x \to \infty$? **2**

(b)



The parabola frog (Frogus projectilius) jumps with initial velocity v m/s at an angle of projection α and its path traces a parabolic arc as shown above.

The frog's horizontal displacement from the origin, t seconds after jumping is given by the equation $x = vt\cos\alpha$ [Do not prove this].

The frog's vertical displacement from the origin, t seconds after jumping is given by the equation $y = vt\sin\alpha - \frac{1}{2}gt^2$ [Do not prove this].

(i) Show that the frog lands after $\frac{2v\sin\alpha}{g}$ seconds **1**

(ii) Show that the frog's range is $\frac{v^2\sin 2\alpha}{g}$ metres **1**

Question 7 continues on the next page

Question 7 (continued)
Marks

(iii) Show that the frog's maximum height is $\dfrac{v^2 \sin^2 \alpha}{2g}$ metres **2**

(iv) Show that the ratio of the frog's maximum height to range is $\dfrac{\tan \alpha}{4}$ **2**

(v) Let $\alpha_{\max}$ be the angle the frog jumps at to ensure maximum range.

Find the ratio of the frog's maximum height to its range in this case. **1**

(vi) Let $g = 10m/s^2$ and $v = \sqrt{85}\, m/s$.

Let α_{equal} be the angle where the frog's maximum height equals its range.

At this angle, the frog needs a 40cm gap to squeeze between a pole and a freeway.

The top of the pole is the focus of the parabola and the freeway is the directrix.

Can the frog squeeze through? **3**

End of Paper

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STANDARD INTEGRALS

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1}, \quad n \neq -1 \; ; \quad x \neq 0, \quad \text{if } n < 0$$

$$\int \frac{1}{x} \, dx = \ln x, \quad x > 0$$

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax}, \quad a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax, \quad a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax, \quad a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1} \frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} \, dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} \, dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE: $\ln x = \log_e x, \quad x > 0$

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① (a) $\frac{x-3}{x-2} - \frac{x-2}{x-2} < 0$

$\frac{x-3-x+2}{x-2} < 0$

$\frac{-1}{x-2} < 0$

$\therefore -(x-2) < 0$ since if $\frac{a}{b} < 0$, $ab < 0$

$\therefore x-2 > 0$

$x > 2$

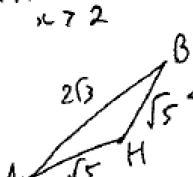
OR $(x-3)(x-2) < (x-2)^2$

$(x-2)\left[x-3-(x-2)\right] < 0$

$(x-2)(-1) < 0$

✓

(b) Base Δ:



$\cos\angle AHB = \frac{5+5-12}{2\sqrt{5}\sqrt{5}}$

$\cos\angle AHB = \frac{-1}{5}$

$\therefore \angle AHB = 101°32'$

(c) (i) $\frac{x-(x+h)}{x(x+h)} = \frac{-h}{x(x+h)}$

(ii) $f'(x) = \lim_{h\to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h}$

$= \lim_{h\to 0} \frac{\frac{-h}{x(x+h)}}{\frac{h}{1}}$

$= \lim_{h\to 0} \frac{-h}{hx(x+h)}$

$= \lim_{h\to 0} \frac{-1}{x(x+h)}$

$= \frac{-1}{x^2}$

(d) $u = 1-x \rightarrow x = 1-u$

$du = -dx$

$x = -3,\ u = 4$

$x = 0,\ u = 1$

$\therefore -\int_4^1 (1-u)\sqrt{u}\, du$

$= \int_1^4 \sqrt{u} - u^{\frac{3}{2}}\, du$

$= \left(\frac{2}{3}u^{\frac{3}{2}} - \frac{2}{5}u^{\frac{5}{2}}\right)_1^4$

$= \frac{2}{3}(8) - \frac{2}{5}(32) - \frac{2}{3} + \frac{2}{5}$

$= -7\frac{11}{15}$

② (a)

$V = \pi\int_{\frac{\pi}{18}}^{\frac{\pi}{9}} \cos^2 3x\, dx$

$\cos 2x = 2\cos^2 x - 1$

$\therefore \cos^2 x = \frac{\cos 2x + 1}{2}$

$\therefore \cos^2 3x = \frac{\cos 6x + 1}{2}$

$\therefore V = \frac{\pi}{2}\int_{\frac{\pi}{18}}^{\frac{\pi}{9}} \cos 6x + 1\, dx$

$= \frac{\pi}{2}\left[\frac{\sin 6x}{6} + x\right]_{\frac{\pi}{18}}^{\frac{\pi}{9}} = \frac{\pi}{2}\left[\frac{\sin\frac{2\pi}{3}}{6} + \frac{\pi}{9} - \frac{\sin\frac{\pi}{3}}{6} - \frac{\pi}{18}\right]$

$= \frac{\pi}{2}\left[\frac{\pi}{18}\right] = \frac{\pi^2}{36}\ u^3$

(b) (i) $f(a) = a^6 - 2a^4 + a^2$

$f(-a) = (-a)^6 - 2(-a)^4 + (-a)^2$

$= a^6 - 2a^4 + a^2$ $\therefore f(a) = f(-a)$ $\therefore$ Even fn

(ii) $(x^2)^2 - 2(x^2) + 1 = (x^2-1)^2$

(iii) $y = 0$ ∴ $0 = x^6 - 2x^4 + x^2$

$0 = x^2(x^4 - 2x^2 + 1)$

$0 = x^2(x^2-1)^2$

$\therefore$ x-int: $0, \pm 1$

(iv) $f'(x) = 6x^5 - 8x^3 + 2x = 0$ For stat pts

$2x(3x^4 - 4x^2 + 1) = 0$

$2x(3x^2-1)(x^2-1) = 0$

$\therefore x = 0, \pm 1, \pm\frac{1}{\sqrt{3}}$

Stat pts $(0,0)$

$(1,0)$ $(-1,0)$

$\left(\frac{1}{\sqrt{3}}, \frac{4}{27}\right)$ $\left(-\frac{1}{\sqrt{3}}, \frac{4}{27}\right)$

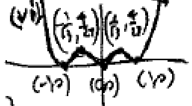
(v) $f''(x) = 30x^4 - 24x^2 + 2$.

$f''(0) = 2$ ∴ Min TP $(0,0)$

$f''(1) = 8$ ∴ Min TP $(1,0)$, $(-1,0)$

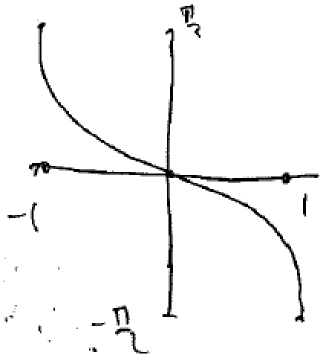
$f''\left(\frac{1}{\sqrt{3}}\right) = -2\frac{2}{3}$ ∴ Max TP $\left(\frac{1}{\sqrt{3}}, \frac{4}{27}\right)$ $\left(-\frac{1}{\sqrt{3}}, \frac{4}{27}\right)$

(vi)



(3)

(a) (i)



(ii) $y' = \cos^{-1}x - \frac{1}{\sqrt{1-x^2}}x - \frac{1}{2\sqrt{1-x^2}}x(-2x)$

$= \cos^{-1}x - \frac{x}{\sqrt{1-x^2}} + \frac{x}{\sqrt{1-x^2}}$

$= \cos^{-1}x$

(iii) $= 0$ since odd fn (see graph)

Proving odd fn:

$f(a) = \cos^{-1}a - \frac{\pi}{2}$

$f(-a) = \cos^{-1}(-a) - \frac{\pi}{2}$

$= \pi - \cos^{-1}(a) - \frac{\pi}{2}$

$= \frac{\pi}{2} - \cos^{-1}a$

$= -f(a)$

OR: $\left[x\cos^{-1}x - \sqrt{1-x^2} - \frac{\pi}{2}x\right]_{-1}^{1}$

$= \cos^{-1}1 - 0 - \frac{\pi}{2} + \cos^{-1}(-1) + 0 - \frac{\pi}{2}$

$= -\pi + \cos^{-1}(-1) = -\pi + \pi = 0$

(b) $t = 0,\ x = 1,\ v = \ln 3$

(i) $\frac{d\frac{1}{2}v^2}{dx} = (\ln 3)^2 x$

$\therefore \frac{1}{2}v^2 = \frac{(\ln 3)^2 x^2}{2} + c$

$v^2 = (\ln 3)^2 x^2 + C$

Now $x = 1,\ v = \ln 3$

$\therefore (\ln 3)^2 = (\ln 3)^2 + c \quad \therefore c = 0$

$\therefore v^2 = (\ln 3)^2 x^2$

$\therefore v = \pm(\ln 3)x$ ↗ but when $x = 1,\ v = \ln 3$ $\therefore v = (\ln 3)x$

(ii) $\frac{dx}{dt} = (\ln 3)x$

$\frac{dt}{dx} = \frac{1}{(\ln 3)x}$

$t = \frac{1}{\ln 3}\ln x + c$

$x = 1,\ t = 0$

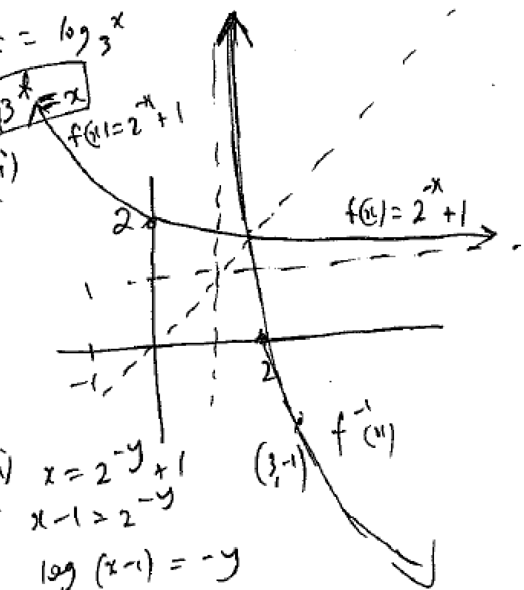
$\therefore 0 = 0 + c \quad \therefore c = 0$

$\therefore t = \frac{\ln x}{\ln 3}$

$t = \log_3 x$

$\therefore 3^t = x$

(4) (a) (i)



(ii) $x = 2^{-y} + 1$

$x - 1 = 2^{-y}$

$\log_2(x-1) = -y$

$\therefore f^{-1}(x) = -\log_2(x-1)$

(iii) See above

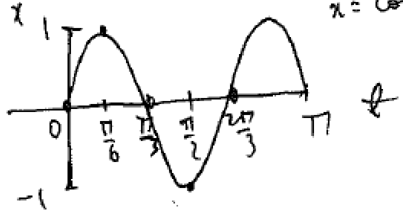
(b)(i) $\dot{x} = -3\sin(3t - \frac{\pi}{2})$

$\ddot{x} = -9\cos(3t - \frac{\pi}{2})$

$= -9x \quad \therefore$ SHM, $n = 3$

$x = \cos\left(3\left(t - \frac{\pi}{6}\right)\right)$

(ii)



(iii) 6 m

(iv) $x = \sin 3t$

(v) $v = 3\cos 3t$

Max speed $= 3$ m/s

$\therefore$ When is speed 1.5 m/s? ↗

$1.5 = 3\cos 3t$

$0.5 = \cos 3t$

$3t = \frac{\pi}{3}$

$t = \frac{\pi}{9}$ sec

(5) (a) (i) $M_{OP} = \frac{ap^2}{2ap} = \frac{p}{2}$

$\therefore M_{perp\,OP} = -\frac{2}{p}$

(ii) $A(2ap, 0)$

(iii) $y - 0 = -\frac{2}{p}(x - 2ap)$

$y = -\frac{2}{p}x + 4a$

(iv) $(0, 4a)$

(b) (i) $(2,1)$ $(8,4)$

$2:1$

$I\left(\frac{2+16}{3}, \frac{1+8}{3}\right) = I(6,3)$

(ii) $2:-1$

$E\left(\frac{-2+16}{1}, \frac{-1+8}{1}\right) = E(14,7)$

(iii) Diameter $(6,3)$ to $(14,7)$

$\therefore$ Centre $(10,5)$

Diameter $= \sqrt{8^2+4^2} = \sqrt{64+16}$
$= \sqrt{80}$
$= 4\sqrt{5}$

$\therefore$ Radius $= 2\sqrt{5}$

$\therefore$ eqn: $(x-10)^2 + (y-5)^2 = (2\sqrt{5})^2$

$(x-10)^2 + (y-5)^2 = 20$

OR: $2\sqrt{(x-8)^2+(y-4)^2} = \sqrt{(x-2)^2+(y-1)^2}$

$4\left[x^2 - 16x + 64 + y^2 - 8y + 16\right] = x^2 - 4x + 4 + y^2 - 2y + 1$

$3x^2 - 60x + 3y^2 - 30y = -315$

$x^2 - 20x + y^2 - 10y = -105$

$x^2 - 20x + 100 + y^2 - 10y + 25 = 20$

$(x-10)^2 + (y-5)^2 = (2\sqrt{5})^2$

(6) (a) (i) $\frac{6!}{2} = 360$

(ii) [EACH]NC $\therefore \frac{3!}{360} = \frac{6}{360} = \frac{1}{60}$

(iii) ${}^6P_3 = 6\times5\times4 = 120$

(iv) Case 1: No Cs $\rightarrow$ HANE $= {}^4P_3 = 24$

Case 2: 1 C $\rightarrow$ (C)HANE $= \binom{4}{2}\times 3! = 36$

Case 3: 2 Cs $\rightarrow$ (CC)HANE $= \binom{4}{1}\times 3 = 12$

$\Downarrow$

$\underline{72}$

(b) $T_6 = \binom{11}{5}\left(\frac{2x}{3}\right)^6\left(\frac{-3}{2x}\right)^5$

$= \binom{11}{5}\left(\frac{64x^6}{3^6}\right)\times -\left(\frac{3^5}{32x^5}\right)$

$= \binom{11}{5}\left(\frac{2x}{3}\right)(-1)$

$= -308x$

(c) For $n=1$, LHS $= a$

RHS $= \frac{1}{2}\left[2a+0\right] = a$

$\therefore$ True for $n=1$.

Let k be int such that

$a + (a+d) + (a+2d) + \dots + (a+(k-1)d) = \frac{k}{2}\left[2a+(k-1)d\right]$

Prove $k+1$ is int such that

$a + (a+d) + (a+2d) + \dots + (a+(k-1)d) + (a+(kd)) = \frac{k+1}{2}\left[2a+kd\right]$

Proof: LHS $= \frac{k}{2}\left[2a+(k-1)d\right] + a + kd$ (from assumption)

$= \frac{k\left[2a+(k-1)d\right] + 2a + 2kd}{2}$

$= \frac{2ak + k^2d - kd + 2a + 2kd}{2}$

$= \frac{2ak + k^2d + 2a + kd}{2}$

$= \frac{2a(k+1) + kd(k+1)}{2}$

$= \frac{k+1}{2}(2a + kd) =$ RHS

$\therefore$ If true for $n=k$, also true for $n=k+1$

$\therefore$ If true for $n=1$, ——— $n=2$
——— $n=3$
& so on $n=3$, all the integer n

(7) (a) $\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt}$

$\frac{dy}{dt} = \frac{12x}{1+(3x^2+5)^2} \times \frac{dx}{dt}$

As $x \to \infty$,

$\therefore \frac{dy}{dt} \to 0 \times \frac{dx}{dt} = 0 \times k = 0$

$\therefore \frac{dy}{dt} \to 0$

(b) (i) $0 = t(v\sin\alpha - \frac{1}{2}gt)$

~~$t = 0$~~ or $v\sin\alpha = \frac{gt}{2}$

$t = \frac{2v\sin\alpha}{g}$

(ii) $x = v\,\frac{2v\sin\alpha}{g}\cos\alpha$

$= \frac{v^2}{g}(2\sin\alpha\cos\alpha)$

$= \frac{v^2}{g} \times \sin 2\alpha$

(iii) $t_{\text{max height}} = \frac{v\sin\alpha}{g}$

$y = \frac{v\,v\sin\alpha}{g}\sin\alpha - \frac{g}{2}\frac{v^2\sin^2\alpha}{g^2}$

$y = \frac{v^2\sin^2\alpha}{g} - \frac{v^2\sin^2\alpha}{2g} = \frac{2v^2\sin^2\alpha}{2g} - \frac{v^2\sin^2\alpha}{2g}$

$= \frac{v^2\sin^2\alpha}{2g}$

(iv) Ratio $= \dfrac{\frac{v^2\sin^2\alpha}{2g}}{\frac{v^2\sin 2\alpha}{g}}$

$= \frac{v^2 g\sin^2\alpha}{2v^2 g\sin 2\alpha} = \frac{\sin^2\alpha}{2\sin 2\alpha}$

$= \frac{\sin^2\alpha}{2 \times 2\sin\alpha\cos\alpha}$

$= \frac{\sin\alpha}{4\cos\alpha} = \frac{\tan\alpha}{4}$

(v) Max range is $\alpha = \frac{\pi}{4}$.

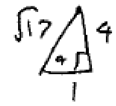
$\therefore$ Ratio $= \frac{\tan\frac{\pi}{4}}{4} = \frac{1}{4}$

(vi) max height = range

$\therefore \frac{\tan\alpha}{4} = 1$

$\therefore \tan\alpha = 4$

$\alpha = \tan^{-1}4 = 75°58'$ (n.m)



$\therefore x = \frac{\sqrt{85}\,t}{\sqrt{17}}$

$x = \sqrt{5}\,t$

$y = \frac{\sqrt{85}\,t \cdot 4}{\sqrt{17}} - 5t^2$

$y = 4\sqrt{5}\,t - 5t^2$

$y = 4x - 5\left(\frac{x}{\sqrt{5}}\right)^2$

$y = 4x - \frac{5x^2}{5}$

$y = 4x - x^2$

$x^2 - 4x = -y$

$x^2 - 4x + 4 = -y + 4$

$(x-2)^2 = -(y-4)$

$\therefore$ Vertex $(2, 4)$

Focal length $4a = 1$

$a = \frac{1}{4}$

$\therefore$ Distance between pole & freeway is $2 \times \frac{1}{4} = \frac{1}{2}$ metre $= 50$cm

$\therefore$ Yes, can squeeze through by 10cm